Rough 3-valued algebras

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Abstract

Rough set theory is an important tool for dealing with granularity and vagueness in information systems. This paper studies a kind of rough set algebra. The collection of all the rough sets of an approximation space can be made into a 3-valued Lukasiewicz algebra. We call the algebra a rough 3-valued Lukasiewicz algebra. In this paper, we focus on the rough 3-valued Lukasiewicz algebras, which are a special kind of 3-valued Lukasiewicz algebras. Firstly, we examine whether the rough 3-valued Lukasiewicz algebra is an axled 3-valued Lukasiewicz algebra. Secondly, we present the condition under which the rough 3-valued Lukasiewicz algebra is also a 3-valued Post algebra. Then we investigate the 3-valued Post subalgebra problem of the rough 3-valued Lukasiewicz algebra. Finally, this paper studies the relationship between the rough 3-valued Lukasiewicz algebra and the Boolean algebra constructed by all the exact sets of the corresponding approximation space.

Keywords: Rough set theory; Approximation space; 3-Valued Lukasiewicz algebra; 3-Valued Post algebra; Boolean algebra

1. Introduction

Rough set theory was introduced by Pawlak [32,33] to account for the definability of a concept in an approximation space \((U, R)\), where \(U\) is a set, and \(R\) is an equivalence relation on \(U\). It captures and formalizes the basic phenomenon of information granulation. The finer the granulation, the more concepts are definable in it. For those concepts not definable in an approximation space, their lower and upper approximations can be defined. These approximations construct a representation of the given concept in the approximation space.

There have been extensive studies on the rough set theory by algebraic methods [1–3,5–31,33–35,37–42]. Wong and Nie [38] and Wong et al. [39], proposed a generalization of the notion of rough sets under interval structures. Lin and Liu [29] replaced Pawlak space with Frechet space, and the equivalence classes are replaced by neighborhoods at the same time. By means of the two replacements, they defined more general approximation operators. Yao [40] interpreted rough set theory as an extension of set theory with two additional unary set-theoretic operators, referred to as approximation operators. Such an interpretation is consistent with
interpreting modal logic as an extension of classical two-valued logic with two added unary operators. Based on atomic Boolean lattice, Jarvinen [25] proposed a more general framework for the study of approximation. Qi and Liu [35] introduced a pair of dual rough operations on Boolean algebras and used them to interpret some uncertainty measures on Boolean algebras. Dai [11] introduced molecular lattices into the research on rough set theory and constructed an even more general structure of rough approximations based on molecular lattices.

Kuroki [26] introduced the notion of a rough ideal in a semigroup. Kuroki and Wang [27] gave some properties of the lower and upper approximations with respect to the normal subgroups. In [16,17], Davvaz was concerned about the relationship between rough sets and ring theory and considered a ring as a universal set and introduced the notion of rough ideals and rough subrings with respect to an ideal of a ring. Also, rough modules have been investigated by Davvaz and Mahdavipour [18]. Corsini [9] showed that how join spaces can be associated with rough sets. Leoreanu [28] introduced a wide class of join spaces as hypergroups associated with rough sets. Also, Corsini and Leoreanu [10] studied fuzzy sets and join spaces associated with rough sets. Zhu studied covering-based rough sets in [42–44].

At the same time, researchers also study rough sets from the perspective of description of the rough set pairs, i.e. (lower approximation set, upper approximation set). Iwinski [23] suggested a lattice-theoretic approach. Iwinski’s aim, which was later extended by Pomykala and Pomykala [34], was to endow the rough sets of \((U, R)\) with a natural algebraic structure. Iwinski [24] also suggested a quite general notion called rough order. However, it seems difficult to obtain decent structure theorems for a set of rough sets under such general conditions. Gehrke and Walker [21] extended the work of Pomykala and Pomykala [34] by proposing a precise structure theorem for the Stone algebra of rough sets, which is in a setting more general than that in [34]. The work of Pomykala and Pomykala [34] was also improved by Comer [8] who noticed that the collection of rough sets of an approximation space is, in fact, a regular double Stone algebra when one introduced another unary operator, i.e. the dual pseudo-complement operator. Comer [8] also showed the reverse result, i.e. every regular double Stone algebra is isomorphic to a subalgebra of the rough set algebra for some approximation space.

Pagliani [30] investigated rough set systems within the framework of Nelson algebras. He showed that for any approximation space, the corresponding rough set system can be given the structure of a semi-simple Nelson algebra. In fact, Pagliani got this result based on the approximation space algebra, which is a Boolean algebra, and a congruence relation on it. The reverse result, i.e. any finite semi-simple Nelson algebra is isomorphic to the rough set system induced by an approximation space can be found in [31]. It should be noticed that Pagliani used the basic assumption of a finite universe. Banerjee and Chakraborty [2] proposed an algebraic structure called rough algebra while studying rough equality within the framework of the modal system \(S_5\). In the subsequent research, Banerjee and Chakraborty [3] noticed that the so-called rough algebra is a particular case of topological quasi-Boolean algebra which was formalized in [37]. In a topological Boolean algebra, the laws of contradiction and exclude middle fail to hold. In fact, it is a quasi-Boolean algebra which has an interior operator. Banerjee and Chakraborty [3] also proposed so-called pre-rough algebra as the successor of the topological quasi-Boolean algebra by adding more conditions. Iturrioz [22] presented a relation between rough sets and 3-valued Lukasiewicz algebras. Pagliani [31] also studied the relationships between rough sets and 3-valued structures. Dai and his coauthors connected some rough algebras with Brouwer-Zadeh lattices and 3-valued Lukasiewicz algebras in [12–14].

We call the algebra constructed by the collection of rough sets of an approximation space a rough 3-valued Lukasiewicz algebra. In fact, the rough 3-valued Lukasiewicz algebras are a special kind of 3-valued Lukasiewicz algebras. However, the study on this special kind of 3-valued Lukasiewicz algebras is deficient.

In this paper, our main aim is to study the rough 3-valued Lukasiewicz algebra itself. Firstly, we investigate whether the rough 3-valued Lukasiewicz algebra is a 3-valued Lukasiewicz algebra is examined. Then we present the representation for the axis of the rough 3-valued Lukasiewicz algebra. After studying the condition under which the rough 3-valued Lukasiewicz algebra is also a 3-valued Post algebra, we consider the 3-valued Post subalgebra problem of the rough 3-valued Lukasiewicz algebra. Finally, this paper investigates the relationship between the rough 3-valued Lukasiewicz algebra and the Boolean algebra constructed by all the exact sets of the corresponding approximation space.
2. Rough 3-valued Lukasiewicz algebra

Let \((U, R)\) be an approximation space, where \(U\) is the universe and \(R\) is an equivalence relation on \(U\). It should be pointed out that the universe \(U\) is not necessarily finite. The equivalence relation generates an equivalence partition on \(U\). The equivalence classes of \(R\) are called elementary sets in the approximation space. For any \(X \subseteq U\), one can characterize \(X\) by a pair of lower and upper approximations [21]:

\[
\underline{X} = \bigcup \{Y \in U/R \mid Y \subseteq X\},
\]

\[
\overline{X} = \bigcup \{Y \in U/R \mid Y \cap X \neq \emptyset\}.
\]

The lower approximation \(\underline{X}\) of \(X\) in the approximation space \((U, R)\) is the union of all the elementary sets contained in \(X\), and the upper approximation \(\overline{X}\) is the union of all the elementary sets which have a non-empty intersection with \(X\). The pair \((\underline{X}, \overline{X})\) is called a rough set. \((\underline{X}, \overline{X})\) is termed exact (also termed definable) if and only if \(\underline{X} = \overline{X}\). In this paper, we denote by \(RS(U)\) the collection of all rough sets of an approximation space \((U, R)\).

Definition 1 [4]. A \((2,2,1,0,0)\) type structure \((L, \vee, \wedge, \circ, 0, 1)\) is a De Morgan algebra if

\[
\begin{align*}
(DM1) \quad & (L, \vee, \wedge, 0, 1) \text{ is a bounded distributive lattice with the least element } 0 \text{ and the greatest element } 1, \\
(DM2) \quad & x^{oo} = x, \\
(DM3) \quad & (x \wedge y)^o = x^o \vee y^o, \\
(DM4) \quad & (x \wedge y)^o = x^o \vee y^o,
\end{align*}
\]

where \(x, y \in L\).

Lemma 1. In an approximation space \((U, R)\), there exists a \(Z \subseteq U\) satisfying \(Z = \underline{X} \cup \overline{X}\) and \(Z = \underline{X} \cup Y\) for any \(X, Y \subseteq U\).

Proof. Straightforward.  

Lemma 2. In an approximation space \((U, R)\), there exists a \(Z \subseteq U\) satisfying \(Z = \underline{X} \cap \overline{X}\) and \(Z = \underline{X} \cap \overline{Y}\) for any \(X, Y \subseteq U\).

Proof. Straightforward.

Lemma 3. In an approximation space \((U, R)\), there exists a \(Z \subseteq U\) satisfying \(Z = (\underline{X})^c\) and \(Z = (\overline{X})^c\) for any \(X \subseteq U\).

Proof. Straightforward.

Theorem 1. Let \((U, R)\) be an approximation space. Then, \(RS(U)\) can be made into a De Morgan algebra denoted by \((RS(U), \vee, \wedge, \circ, \langle \phi, \phi \rangle, \langle U, U \rangle)\), where \(\langle \phi, \phi \rangle\) is the least element and \(\langle U, U \rangle\) is the greatest element. The operators \(\oplus, \otimes\) and \(*\) are defined as follows:

\[
\begin{align*}
\langle \underline{X}, \overline{Y} \rangle \oplus \langle \underline{Y}, \overline{X} \rangle &= \langle \underline{X} \cup Y, \overline{X} \cup \overline{Y} \rangle, \\
\langle \underline{X}, \overline{Y} \rangle \otimes \langle \underline{Y}, \overline{X} \rangle &= \langle \underline{X} \cap Y, \overline{X} \cap \overline{Y} \rangle, \\
\langle \underline{X}, \overline{Y} \rangle * &= \langle U - \overline{X}, U - \overline{Y} \rangle = \langle (\overline{X})^c, (\overline{Y})^c \rangle.
\end{align*}
\]

Proof. From Lemmas 1–3, we know that the two binary operators \(\otimes, \oplus\) and the unary operator \(*\) are closed operators on \(RS(U)\). We can get (DM1) easily. Here, we just prove (DM2) and (DM3). (DM4) can be proved similarly to (DM3).
(DM2). Let $x = \langle X, \overline{X} \rangle \in RS(U)$. Then, by Eq. (5), we know that $x^* = \langle (X)^c, (X)^c \rangle^* = \langle X, \overline{X} \rangle = x$.

(DM3). Let $x = \langle X, \overline{X} \rangle, y = \langle Y, \overline{Y} \rangle \in RS(U)$. Then, we have $(x \otimes y)^* = (\langle X \cap Y, X \cap \overline{Y} \rangle)^* = (\langle X \rangle^c \cup \langle Y \rangle^c, \langle X \rangle^c \cup \langle Y \rangle^c)^* = x^* \otimes y^*$.

\[ \Box \]

**Definition 2** ([4,36]). A $(2,2,1,0,1,0)$ type structure $(L, \vee, \wedge, \circ, \psi, 0, 1)$ is a 3-valued Lukasiewicz algebra if

(LM1) $(L, \vee, \wedge, \circ, 0, 1)$ is a De Morgan algebra,
(LM2) $\psi(x \wedge y) = \psi(x) \wedge \psi(y)$,
(LM3) $\psi(x \lor y) = \psi(x) \lor \psi(y)$,
(LM4) $\psi(x) \wedge (\psi(x))^\circ = 0$,
(LM5) $\psi(\psi(x)) = \psi(x)$,
(LM6) $\psi((\psi(x))^\circ) = (\psi(x))^\circ$,
(LM7) $(\psi(x^\circ))^\circ \leq \psi(x)$,
(LM8) $\psi(x) = \psi(y), \psi(x^\circ) = y^\circ$ imply $x = y$,

where $x, y \in L$.

Let us introduce another unary operator $\triangle$ into the rough De Morgan algebra $(RS(U), \oplus, \otimes, *, \langle \phi, \phi \rangle, \langle U, U \rangle)$ by the following method:

$$\langle X, \overline{X} \rangle^\triangle = \langle X, \overline{X} \rangle.$$  \hspace{1cm} (6)

Now we come to the important result, i.e. the collection of all the rough sets of an approximation space can be made into a 3-valued Lukasiewicz algebra. The result can be found in [22,31]. The proof was omitted in [22]. While in [31], the proof was completed in finite case based on a Boolean algebra and a congruence relation on it. In the following, we complete the proof based on approximation sets pairs method.

**Theorem 2.** Let $(U, R)$ be an approximation space. Then, $RS(U)$ can be made into a 3-valued Lukasiewicz algebra denoted by $(RS(U), \oplus, \otimes, *, \triangle, \langle \phi, \phi \rangle, \langle U, U \rangle)$, where $\langle \phi, \phi \rangle$ is the least element and $\langle U, U \rangle$ is the greatest element. The operators $\oplus, \otimes, *$ and $\triangle$ are defined by Eqs. (3)–(6).

**Proof.** We can get (LM1) easily by Theorem 1.

(LM2). We need to prove $(x \wedge y)^\triangle = x^\triangle \wedge y^\triangle$. Let $x = \langle X, \overline{X} \rangle, y = \langle Y, \overline{Y} \rangle$. Then, by Eq. (6), we know that $(x \wedge y)^\triangle = \langle X \cap Y, \overline{X} \cap Y \rangle^\triangle = \langle \overline{X}, \overline{Y} \rangle = x$.

(LM3). It is similar to the proof of (LM2).

(LM4). We need to prove $x^\triangle \wedge x^{\circ \triangle} = \langle \phi, \phi \rangle$. Let $x = \langle X, \overline{X} \rangle$. Then, we know that $x^\triangle = \langle \overline{X}, \overline{X} \rangle$ and $x^{\circ \triangle} = \langle (X)^c, (X)^c \rangle^\circ = \langle \overline{X}, \overline{X} \rangle \circ$. Consequently, it is easy to obtain $x^\triangle \wedge x^{\circ \triangle} = \langle \phi, \phi \rangle$.

(LM5). We need to prove $x^{\circ \triangle \triangle} = x^\triangle$. Let $x = \langle X, \overline{X} \rangle$. Then, we know that $x^\triangle = \langle \overline{X}, \overline{X} \rangle$. Hence, $x^{\circ \triangle \triangle} = \langle \overline{X}, \overline{X} \rangle^\triangle = \langle X, \overline{X} \rangle = x^\triangle$.

(LM6). We need to prove $x^{\circ \triangle \triangle} = x^\triangle$. Let $x = \langle X, \overline{X} \rangle$, then we have $x^\triangle = \langle (X)^c, (X)^c \rangle$.

(LM7). We need to prove $x^{\circ \triangle \triangle} = x^\triangle$. Let $x = \langle X, \overline{X} \rangle$, then we know $x^{\circ \triangle \triangle} = \langle (X)^c, (X)^c \rangle^{\circ \triangle \triangle} = \langle (X)^c, (X)^c \rangle^\circ = \langle X, \overline{X} \rangle = x^\triangle$.

(LM8). We need to prove that if $x^\triangle = y^\triangle$ and $x^{\circ \triangle} = y^{\circ \triangle}$, then $x = y$. Let $x = \langle X, \overline{X} \rangle, y = \langle Y, \overline{Y} \rangle$. Then, $x^\triangle = y^\triangle$ follows that $X = Y, \overline{X} = \overline{Y}$, i.e. $\overline{X} = \overline{Y}$. $x^{\circ \triangle} = y^{\circ \triangle}$ follows that $\langle (X)^c, (X)^c \rangle = \langle (Y)^c, (Y)^c \rangle$, i.e. $\langle (X)^c, (X)^c \rangle = \langle (Y)^c, (Y)^c \rangle$. It implies $X = Y$. Therefore, $x = y$. \hspace{1cm} \Box

**Definition 3.** The algebra $(RS(U), \oplus, \otimes, *, \triangle, \langle \phi, \phi \rangle, \langle U, U \rangle)$ constructed in Theorem 2 is called the rough 3-valued Lukasiewicz algebra corresponding to the approximation space $(U, R)$. 
3. The axis of rough 3-valued Łukasiewicz algebra

**Definition 4** [4]. A 3-valued Łukasiewicz algebra \((L, \vee, \wedge, \circ, \psi, 0, 1)\) is said to be axled provided there exists an \(a \in L\) such that

\[
\begin{align*}
(A1) & \quad (\psi(a^c))^\circ = 0, \\
(A2) & \quad \psi(x) \leq \psi(a) \lor (\psi(x^c))^\circ \quad (\forall x \in L)
\end{align*}
\]

in which case \(a\) is called the axis of \(L\).

**Definition 5.** In an approximation space \((U, R)\), the Singleton Object Set of \(U\) relative to \(R\) is defined as

\[
S_R(U) = \{ u \in U \mid \text{card}([u]_R) = 1 \}
\]

\((7)\)

\(S_R(U)\) is also denoted by \(S\) if there is no risk of confusion. Dually, the Group Object Set of \(U\) relative to \(R\) is defined as

\[
G_R(U) = \{ u \in U \mid \text{card}([u]_R)) = 1 \}
\]

\((8)\)

\(G_R(U)\) is also denoted by \(G\) if there is no risk of confusion.

**Remark 1.** Both \(S\) and \(G\) can be empty. \(S = \phi\) means that there is no any singleton elementary class in the partition created by the equivalence relation \(R\). \(G = \phi\) means that all the equivalence classes in the partition created by the equivalence relation \(R\) are singleton elementary classes.

**Example 1.** Let \(U = \{a, b, c\}\), \(U/R = \{\{a\}, \{b, c\}\}\), then we have \(S = \{a\}\) and \(G = \{b, c\}\).

**Lemma 4.** Given an approximation space \((U, R)\), let \(S\) be the singleton object set of \(U\) relative to \(R\). Here, \(S \neq \phi\). Let \(X \subseteq U\) and \(s \in S\), then the following are equivalent:

\[\begin{align*}
(a) & \quad s \in X, \\
(b) & \quad s \in \overline{X}, \\
(c) & \quad s \in \overline{X}.
\end{align*}\]

**Proof.** Since \(s \in S\), we know \([s]_R = \{s\}\). Then, by Eqs. (1) and (2), we can prove the lemma. \(\Box\)

**Theorem 3.** Given an approximation space \((U, R)\), let \(G\) be the group object set of \(U\) relative to \(R\). Then, for \(\forall X \subseteq U\), \(X \subseteq \overline{X} \cup G\) holds.

**Proof.** If \(G = \phi\), then \(X \subseteq S = U\). By **Lemma 4**, we know \(X = \overline{X}\). It follows that \(X \subseteq \overline{X} \cup G\). If \(S = \phi\), then \(X \subseteq G = U\). Clearly, \(X \subseteq \overline{X} \cup G\). So we suppose that \(S \neq \phi\) and \(G \neq \phi\) in the following. Suppose the equivalence partition on \(U\) generated by the equivalence relation \(R\) is \(\{G_1, G_2, \ldots, G_P, S_1, S_2, \ldots, S_Q\}\), where \(\text{card}(G_i) = 1\) \((i = 1, 2, \ldots, P)\), \(\text{card}(S_j) = 1\) \((j = 1, 2, \ldots, Q)\), \(\bigcup_{i=1}^{P} S_i = S\) and \(G = \bigcup_{i=1}^{P} G_i\).

\[
(1) \text{ In case of } \overline{X} \subseteq S, \text{ by **Lemma 4** we have } X = \overline{X} = \overline{X}. \text{ It is obvious that } \overline{X} \subseteq \overline{X} \cup G.
\]

\[
(2) \text{ In case of } \overline{X} \subseteq G, \text{ we have } \overline{X} \subseteq \overline{X} \cup G \text{ directly.}
\]

\[
(3) \text{ Otherwise, } \overline{X} = \bigcup_{i=1}^{P} G_i \bigcup_{j=1}^{Q} S_j, \text{ where } P, Q \leq Q. \text{ It is obvious that } \bigcup_{i=1}^{P} G_i \subseteq \bigcup_{i=1}^{P} G_i = G. \text{ By **Lemma 4**, we can obtain that } \bigcup_{i=1}^{P} S_i \subseteq \overline{X}. \text{ Then, we have } \overline{X} = \bigcup_{i=1}^{P} G_i \bigcup_{j=1}^{Q} S_j \subseteq \overline{X} \cup G.
\]

From (1)–(3) above, we know that this theorem holds. \(\Box\)

**Theorem 4.** The rough 3-valued Łukasiewicz algebra \((\text{RS}(U), \oplus, \otimes, \ast, \langle \phi, \phi \rangle, (U, U))\) is necessarily axled and the axis is \(\langle \phi, G \rangle\), where \(G\) is the group object set.
Proof. (1) We first prove that $RS(U)$ is axled. It is straightforward when $G = \phi$. So we just consider $G \neq \phi$. If $S \neq \phi$, then the equivalence partition is \{\{G_1, G_2, \ldots, G_P, S_1, S_2, \ldots, S_Q\}\}, where card \((G_i)\) \(i = 1, 2, \ldots, P\), card \((S_j)\) \(j = 1, 2, \ldots, Q\), \(\bigcup_{i=1}^{P} S_i = S\) and \(\bigcup_{i=1}^{P} G_i = G\). Otherwise, $S = \phi$, then the equivalence partition on $U$ created by the equivalence relation $R$ is \{\{G_1, G_2, \ldots, G_P\}\}. In both cases, we take $X = \{g_1, g_2, \ldots, g_P\}$ where $g_i$ is an object in $U$ and $g_i \in G_i$ \((i = 1, 2, \ldots, P)\). By Eq. (1), we obtain $X = \{x \in U \mid [x]_R \subseteq X\} = \phi$. By Eq. (2), we have $X = \{x \in U \mid [x]_R \cap X \neq \phi\} = \bigcup_{i=1}^{P} G_i = G$. From above, we can conclude that $\langle \phi, G \rangle \in RS(U)$.

(2) We then prove that $\langle \phi, G \rangle$ is the axis of the algebra. Let $a = \langle \phi, G \rangle$, then $a^* = \langle S, U \rangle = \langle U, U \rangle^* = \langle \phi, \phi \rangle$. So, condition (A1) in Definition 4 is satisfied. At the same time, $x^* = \langle X, X \rangle = \langle G, G \rangle = \langle X \cup G, X \cup G \rangle$. Additionally, $x^* = \langle X, X \rangle$. By Theorem 3, we know that $x^* = \langle x, x \rangle$. It means that condition (A2) in Definition 4 holds. By Definition 4, we can conclude that $\langle \phi, G \rangle$ is the axis. □

From (1) and (2) above, we can prove the theorem.

Example 2. Let $U = \{a, b, c\}$, $U/R = \{\{a\}, \{b, c\}\}$, then we know that the axis of $RS(U)$ is $\langle \phi, \{b, c\} \rangle$.

Corollary 1. Let $(U, R)$ be an approximation space, then the corresponding rough 3-valued Lukasiewicz algebra $(RS(U), \oplus, \ominus, \ast, A, \langle \phi, \phi \rangle, \langle U, U \rangle)$ has the axis $\langle \phi, \phi \rangle$ if and only if $G = \phi$.

Example 3. Let $U = \{a, b, c\}$, $U/R = \{\{a\}, \{b\}, \{c\}\}$, then we know that the axis of $RS(U)$ is $\langle \phi, \phi \rangle$.

4. From rough 3-valued Lukasiewicz algebra to 3-valued Post algebra

Definition 6 [4]. A 3-valued Lukasiewicz algebra $(L, \lor, \land, \circ, \psi, 0, 1)$ is said to be centered if and only if there exists a $c \in L$, such that

\[
\begin{align*}
(C1) & \quad \psi(c) = 1, \\
(C2) & \quad (\psi(c^*))^* = 0,
\end{align*}
\]

in which case $c$ is called the center of $L$.

Theorem 5. The rough 3-valued Lukasiewicz algebra $(RS(U), \oplus, \ominus, \ast, A, \langle \phi, \phi \rangle, \langle U, U \rangle)$ is centered if and only if $S = \phi$, where $S$ is the singleton object set. The center is $\langle \phi, \phi \rangle$ if it exists.

Proof

(\(\Leftarrow\)) We first prove that $(RS(U), \oplus, \ominus, \ast, A, \langle \phi, \phi \rangle, \langle U, U \rangle)$ is centered if $S = \phi$. Since $S = \phi$, we can set $U/R = \{G_1, G_2, \ldots, G_P\}$, in which card \((G_i)\) \(i = 1, 2, \ldots, P\). Here $\bar{G} = \bigcup_{i=1}^{P} G_i = U$. We take $X = \{g_1, g_2, \ldots, g_P\}$ where $g_i$ is an object in $U$ and $g_i \in G_i$ \((i = 1, 2, \ldots, P)\). Then, $X = \{x \in U \mid [x]_R \subseteq X\} = \{x \in U \mid x \in G_i \subseteq X, i = 1, 2, \ldots, P\} = \phi$. At the same time, $\bar{X} = \{x \in U \mid [x]_R \cap X \neq \phi\} = \bigcup_{i=1}^{P} G_i = U$. It follows that $\langle \phi, U \rangle \in RS(U)$. It is easy to obtain that $\langle \phi, U \rangle^\ast = \langle \bar{U}, \bar{U} \rangle$. Additionally, we can have $\langle \phi, U \rangle^\ast = \langle \bar{U}, \bar{U} \rangle^\ast = \langle \phi, \phi \rangle$. By Definition 6, we know that $(RS(U), \oplus, \ominus, \ast, A, \langle \phi, \phi \rangle, \langle U, U \rangle)$ is centered. And the center is $\langle \phi, \phi \rangle$.

(\(\Rightarrow\)) We need to prove $S = \phi$ if $(RS(U), \oplus, \ominus, \ast, A, \langle \phi, \phi \rangle, \langle U, U \rangle)$ is centered. Since $(RS(U), \oplus, \ominus, \ast, A, \langle \phi, \phi \rangle, \langle U, U \rangle)$ is centered, we can set the center $c = \langle \mathcal{C}, \mathcal{C} \rangle \in RS(U)$, then by Definition 6 we can get $\langle \mathcal{C}, \mathcal{C} \rangle = \langle \mathcal{C}, \mathcal{C} \rangle = \langle U, U \rangle$, i.e. \(\bar{C} = U\). At the same time, $\langle \mathcal{C}, \mathcal{C} \rangle^\ast = \langle \mathcal{C}, \mathcal{C} \rangle = \langle \phi, \phi \rangle$, i.e. \(\bar{C} = U\). Then, we have $c = \langle \mathcal{C}, \mathcal{C} \rangle = \langle \phi, \phi \rangle$. It follows that $\langle \phi, U \rangle \in RS(U)$. Namely, there must exist an $X \subseteq U$ such that $X = \phi$ and $\bar{X} = \bar{U}$. Now, we suppose that $S \neq \phi$. Then, there exists an $s \in U$ such that $[s]_R = \{s\}$. Since $\bar{X} = \bar{U}$, then $s \in \bar{X}$, by Lemma 4 we have $s \in X$. On the other hand, $X = \phi$, which implies that $s \notin X$. By Lemma 4, we obtain $s \notin X$. This is a contradiction. Thus, $S = \phi$. □
It should be pointed out that Iturrioz [22] mentioned that the center (if it exists) of the rough 3-valued Lukasiewicz algebra is the element $\langle \phi, U \rangle$. Pagliani [31] also mentioned that the center is $\langle \phi, \phi \rangle$ in his disjoint representation.

**Example 4.** Let $U = \{a, b, c\}, U/R = \{\{a\}, \{b, c\}\}$, then we know that the corresponding rough 3-valued Lukasiewicz algebra $RS(U)$ is not centered.

**Example 5.** Let $U = \{a, b, c, d\}, U/R = \{\{a\}, \{b, c\}, \{d\}\}$, then we know that the corresponding rough 3-valued Lukasiewicz algebra $RS(U)$ is centered, and the center is $\langle \phi, U \rangle$.

**Theorem 6.** If the rough 3-valued Lukasiewicz algebra $(RS(U), \oplus, \otimes, *, \Delta, \langle \phi, \phi \rangle, \langle U, U \rangle)$ is centered, then the center is the axis.

**Proof.** It is straightforward from Theorems 4 and 5. $\Box$

**Example 6.** Let $U = \{a, b, c, d\}, U/R = \{\{a\}, \{b, c\}, \{d\}\}$. Then, we know $G = U = \{a, b, c, d\}$. It follows that the axis of the corresponding rough 3-valued Lukasiewicz algebra $RS(U)$ is $\langle \phi, G \rangle = \langle \phi, U \rangle$. From Example 5, we know that the axis is also the center.

**Theorem 7.** For approximation space $(U, R)$, suppose that the corresponding rough 3-valued Lukasiewicz algebra is $(RS(U), \oplus, \otimes, *, \Delta, \langle \phi, \phi \rangle, \langle U, U \rangle)$, and $G$ is the group object set, where $G \neq \phi$. Then, $(RS(G), \oplus, \otimes, *, \Delta, \langle \phi, \phi \rangle, \langle G, G \rangle)$ is a centered 3-valued Lukasiewicz algebra (or a 3-valued Post algebra), where $\langle \phi, \phi \rangle$ is the least element and $\langle G, G \rangle$ is the greatest element. The operators $\oplus, \otimes$ and $\Delta$ are defined by Eqs. (3), (4) and (6), respectively. The unary operator $+^*$ is defined as

$$\langle X, X \rangle^* = \langle G - X, G - X \rangle$$

(9)

The center is $\langle \phi, G \rangle$.

**Proof.** This theorem can be proved in the similar way to Theorem 5. $\Box$

**Definition 7.** The algebra $(RS(G), \oplus, \otimes, +^*, \Delta, \langle \phi, \phi \rangle, \langle G, G \rangle)$ constructed in Theorem 7 is called a group object rough 3-valued Lukasiewicz algebra.

**Definition 8.** In the rough 3-valued Lukasiewicz algebra $(RS(U), \oplus, \otimes, *, \Delta, \langle \phi, \phi \rangle, \langle U, U \rangle)$, by $[x]$ we denote the set $\{y \in RS(U) | y \leq x\}$, where $x \in RS(U)$. $[x]$ is an ideal on $RS(U)$, called the principal ideal generated by $x$.

**Remark 2.** For $x, y \in RS(U)$, let $x = \langle X, \bar{X} \rangle$, $y = \langle Y, \bar{Y} \rangle$. $y \leq x$ if and only if $\langle X, \bar{X} \rangle \oplus \langle Y, \bar{Y} \rangle = \langle X, \bar{X} \rangle$, i.e. $\bar{Y} \subseteq \bar{X}$ and $\bar{Y} \subseteq \bar{X}$.

**Theorem 8.** Let $(U, R)$ be an approximation space. Suppose that the group object set $G \neq \phi$. Suppose that the corresponding rough 3-valued Lukasiewicz algebra is $(RS(U), \oplus, \otimes, *, \Delta, \langle \phi, \phi \rangle, \langle U, U \rangle)$, and the axis is $a = \langle \phi, G \rangle$. Then, $(a^*, \oplus, \otimes, +^*, \Delta, \langle \phi, \phi \rangle, a^*)$ is an algebra, and it is just the group object rough 3-valued Lukasiewicz algebra $RS(G)$.

**Proof.**

1. We first prove that the operators $\oplus, \otimes, +^*, \Delta$ are closed. Since $a^* = \langle G, G \rangle$, then by Eqs. (3), (4), (6) and (9) we can prove the four operators are closed in $\langle a^* \rangle$. It follows that $\langle a^* \rangle$ is an algebra.

2. We then prove $\langle a^* \rangle = RS(G)$. $\langle a^* \rangle = \{\langle X, \bar{X} \rangle | X \subseteq U, \langle X, \bar{X} \rangle \leq a^*\}$. $RS(G) = \{\langle X, \bar{X} \rangle | X \subseteq G\}$. For any arbitrary $X \subseteq G$, it is obvious that $\langle X, \bar{X} \rangle \leq \langle G, G \rangle = a^*$, then we have $RS(G) \subseteq \langle a^* \rangle$. For any $X \subseteq U$ such that $\langle X, \bar{X} \rangle \leq a^* = \langle G, G \rangle$, we suppose that there exists an $s \in X$ such that $s \in S = G^c$, where $S$ is
the singleton object set. By Lemma 4, we have \( s \in X \) and \( s \in \overline{X} \). Hence, \( X \not\subseteq G \). Consequently, \( \langle X, \overline{X} \rangle \leq \langle G, G \rangle \) does not hold. This is a contradiction. Thus, for any \( X \subseteq U \) satisfying \( \langle X, \overline{X} \rangle \leq a^d = \langle G, G \rangle \), we have \( X \subseteq G \). It implies that \( a^d \subseteq RS(G) \). From the above, we have \( (a^d) = RS(G) \). On the other hand, the operators are the same. Therefore, we can conclude that the two algebras are identical. \( \square \)

**Theorem 9.** Let \( (U, R) \) be an approximation space. Suppose that the group object set \( G \neq \phi \) and the corresponding rough 3-valued Lukasiewicz algebra is \( (RS(U), \oplus, \otimes, \ast, \Delta, \langle \phi, \phi \rangle, \langle U, U \rangle) \). Then, \( (\langle a^d \rangle, \oplus, \otimes, \ast, \Delta, \langle \phi, \phi \rangle, (a^d)) \) is a centered 3-valued Lukasiewicz algebra (or a 3-valued Post algebra).

**Proof.** It comes from Theorems 7 and 8. \( \square \)

**Example 7.** Let \( U = \{a,b,c\}, U/R = \{\{a\}, \{b,c\}\} \). Then, we know the corresponding rough 3-valued Lukasiewicz algebra \( RS(U) \) is not centered. Its axis is \( \langle \phi, \{b,c\} \rangle \) and \( G = \{b,c\} \). We can construct a centered 3-valued Lukasiewicz algebra (or a 3-valued Post algebra) denoted as \( (\{\{b,c\}, \{b,c\}\}) \), which is identical to \( RS(G) \). This is shown in Fig. 1.

5. From rough 3-valued Lukasiewicz algebra to Boolean algebra

**Definition 9.** In the rough 3-valued Lukasiewicz algebra \( (RS(U), \oplus, \otimes, \ast, \Delta, \langle \phi, \phi \rangle, \langle U, U \rangle) \), we denote by \( C(RS(U)) \) the set of all complemented elements of \( RS(U) \).

**Theorem 10.** Let \( (U, R) \) be an approximation space and \( (RS(U), \oplus, \otimes, \ast, \Delta, \langle \phi, \phi \rangle, \langle U, U \rangle) \) be the corresponding rough 3-valued Lukasiewicz algebra. Then, the algebra \( (C(RS(U)), \oplus, \otimes, \ast, \langle \phi, \phi \rangle, \langle U, U \rangle) \) is a Boolean algebra.

**Proof.** Since a Boolean algebra is a bounded distributive lattice in which all the elements are complemented. So, this theorem is straightforward. \( \square \)

**Definition 10.** In an approximation space \( (U, R) \), we denote by \( E(U) \) the set of all exact sets \( \{\langle X, \overline{X} \rangle \mid X \subseteq U, X = \overline{X} \} \).

**Theorem 11.** Let \( (U, R) \) be an approximation space and \( (RS(U), \oplus, \otimes, \ast, \Delta, \langle \phi, \phi \rangle, \langle U, U \rangle) \) be the corresponding rough 3-valued Lukasiewicz algebra. Then, the algebra \( (E(U), \oplus, \otimes, \ast, \Delta, \langle \phi, \phi \rangle, \langle U, U \rangle) \) and the algebra \( (C(RS(U)), \oplus, \otimes, \ast, \Delta, \langle \phi, \phi \rangle, \langle U, U \rangle) \) are identical.

**Proof.** We only need to prove \( C(RS(U)) = E(U) \).

![Diagram](image-url)

*Fig. 1.* Construct a 3-valued centered Lukasiewicz algebra (a 3-valued Post algebra) from the rough 3-valued Lukasiewicz algebra given in Example 7. The axis \( a \) of the left \( RS(U) \) is \( \langle \phi, \{b,c\} \rangle \) with dotted frame. It becomes the center of the right algebra \( a^d \) (or algebra \( RS(G) \)). The element \( a^d \) of \( RS(U) \), emphasized with a heavy frame, becomes the greatest element of algebra \( a^d \).
Fig. 2. The isomorphism declared in Theorem 12 (or Theorem 13). The dashed lines show the isomorphic connections between the left $RS(U)$ and the right $E(U)$ (or $C(RS(U))$).

(1) For any $x \in E(U)$, suppose that $x = \langle X, \overline{X} \rangle$, then we know $X = X = \overline{X}$. It is obvious that there exists $\langle X^c, \overline{X}^c \rangle \in RS(U)$ such that $\langle X^c, \overline{X}^c \rangle$ is the complement of $x = \langle X, \overline{X} \rangle$. Therefore, $E(U) \subseteq C(RS(U))$.

(2) Then, we need to prove $C(RS(U)) \subseteq E(U)$. For any $x = \langle X, \overline{X} \rangle \in C(RS(U))$, suppose that the complement of $x$ is $y = \langle Y, \overline{Y} \rangle$. Then, we have

$$\langle X, \overline{X} \rangle \otimes \langle Y, \overline{Y} \rangle = \langle X \cap Y, \overline{X \cap Y} \rangle = \langle \phi, \phi \rangle,$$

$$\langle X, \overline{X} \rangle \otimes \langle Y, \overline{Y} \rangle = \langle X \cup Y, \overline{X \cup Y} \rangle = \langle U, U \rangle.$$ 

They imply $X \cap Y = \phi$ and $X \cup Y = U$, i.e. $Y = (X)^c$. Similarly, $\overline{Y} = (\overline{X})^c$. Therefore, $(X)^c \subseteq (X)^c$, i.e. $\overline{X} \subseteq X$. From Eqs. (1) and (2), we have $X \subseteq \overline{X}$. Therefore, $X = \overline{X}$. It follows that $x \in E(U)$, which proves $C(RS(U)) \subseteq E(U)$. □

**Definition 11.** In the rough 3-valued Lukasiewicz algebra $(RS(U), \oplus, \otimes, *, \langle \phi, \phi \rangle, \langle U, U \rangle)$, by $[x]$ we denote the set $\{y \in RS(U) | x \leq y\}$, where $x \in RS(U)$. $[x]$ is a filter on $RS(U)$, called the principal filter generated by $x$. 

**Theorem 12.** Let $(U, R)$ be an approximation space and $(RS(U), \oplus, \otimes, *, \langle \phi, \phi \rangle, \langle U, U \rangle)$ be the corresponding rough 3-valued Lukasiewicz algebra. Suppose that the axis is $a = \langle \phi, G \rangle$, where $G$ is the group object set. Then, $[a]$ is isomorphic to the Boolean algebra $E(U)$, i.e. $[a] \cong E(U)$.

**Proof.** We now define a mapping $f: [a] \to E(U)$ by $f(x) = x^*a^*$. Clearly, $f$ is a surjective morphism. Then, we prove that the mapping is also injective. Suppose that $f(x) = f(y)$ for $x, y \in [a]$, then we have $x^a \geq a^a$ and $x^a \geq x^*a^*$. It follows that $x^a \geq a^a \vee x^*a^* = a^a \vee f(x) = a^a \vee y^*a^*$. Since $a$ is the axis, we have $a^a \vee y^*a^* \geq y^a$. Hence, $x^a \geq y^a$. Similarly, $y^a \geq x^a$. Therefore, $x = y$. □

**Example 8.** Let $U = \{a, b, c\}, U/R = \{[a], [b, c]\}$. Then, we can show the isomorphism by Fig. 2.

**Theorem 13.** Let $(U, R)$ be an approximation space. Suppose that the group object set $G \neq \phi$. Suppose that the corresponding rough 3-valued Lukasiewicz algebra is $(RS(U), \oplus, \otimes, *, \langle \phi, \phi \rangle, \langle U, U \rangle)$, and the axis is $a = \langle \phi, G \rangle$. Then, $[a]$ is isomorphic to $C(RS(U))$, i.e. $[a] \cong C(RS(U))$.

**Proof.** It comes from Theorems 11 and 12. □

6. Conclusions

In this paper, we have studied rough 3-valued Lukasiewicz algebra from three main aspects. Firstly, we have discussed the axis problem of rough 3-valued Lukasiewicz algebra. It is shown that rough 3-valued Lukasiewicz algebra is a 3-valued Lukasiewicz algebra with axis, which is hereof pointed out. It should be noted that the universe $U$ in this paper is not necessarily finite.
Secondly, we have investigated the relationship between 3-valued Lukasiewicz algebra and 3-valued Post algebra. The condition under which rough 3-valued Lukasiewicz algebra is a 3-valued Post algebra is shown in details. It is also revealed that the center is the axis if it does exist. Furthermore, we have illustrated that one can construct a 3-valued Post subalgebra from a rough 3-valued Lukasiewicz algebra, except in a very special case that each elementary class in the approximation space is singleton.

Finally, this paper studied the relationship between rough 3-valued Lukasiewicz algebra and the Boolean algebra constructed by the exact sets of the corresponding approximation space. All the complemented elements of a rough 3-valued Lukasiewicz algebra construct a Boolean subalgebra. At the same time, it is well known that all the exact sets of an approximation space also construct a Boolean algebra. This paper shows that an element in the rough 3-valued Lukasiewicz algebra is complemented, if and only if, it is an exact set (or definable set) in the corresponding approximation space. Thus, the two Boolean algebras become the same one. Additionally, it is shown that the principal filter generated by the axis of the rough 3-valued Lukasiewicz algebra is isomorphic to the two Boolean algebras.

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References