Estimating the stability region of singular perturbation power systems with saturation nonlinearities: an linear matrix inequality-based method

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Abstract: For power systems with detailed excitation and power system stabiliser (PSS) controller, the authors set up a singular perturbation dynamical model with saturation non-linearities for their small-signal stability analysis. To estimate the stability region of this kind of dynamical systems, they establish a set of conditions under which the stability region can be decomposed into the Cartesian product of a contractive high-dimensional ellipse and a sufficiently large set. A method is provided for estimating the stability region with least conservativeness by introducing an linear matrix inequality-based optimisation model. A simulation on a simple power system is described as well.

1 Introduction

In this paper, we consider the problem of the stability region estimation for singular perturbation systems with saturation non-linearities and its application to power systems. Our interest in those systems was motivated in part by their important applications in power systems and many other engineering systems; and in part by the mathematical challenges that arise in extending the results on normal singular perturbation systems to singular perturbation systems with saturation non-linearities. Indeed, the dynamical models for the small-signal stability analysis of power systems with power system stabiliser (PSS) are singular perturbation dynamical models of high dimensions and some saturation non-linearities are existent in these models [1–3]. It is very important to analyse the power systems stability since we want to know whether the systems can withstand some disturbances [3–5]. Estimating the stability region of those systems is thus of fundamental importance.

As we know, the singular perturbation systems can be treated as the special dynamical systems with saturation non-linearities, so some methods which are useful for the stability region estimation of normal dynamical systems with saturation non-linearities, such as the methods provided in [6–11] and the references therein, can also be used for such special dynamical systems. In these works, Hu et al. introduce a promising method in which some interesting ways were provided to deal with the saturation term and depict its properties in terms of linear matrix inequality (LMI). Further, an optimisation model is introduced to seek Lyapunov functions to improve the estimated stability region solution [12]. We have applied this idea to power systems to quantitatively evaluate the performance of PSS [5, 13], opening a new way to analyse the impact of control limits on power system stability. However, the singularly perturbed characteristics are responsible for some numerical problems, so directly estimating the stability region by traditional methods will meet with some numerical problems such as ill-condition and dimension disaster.

Singular perturbation method is a useful tool for the dynamic systems with multi-time scale including the power systems [14–19] and the reference therein. In particular,
Garcia et al. [17, 20] provided a useful method to stabilise singular perturbation systems and under some conditions they also provided a method to estimate the stability region of closed-loop systems. Similar work was done by Liu [21] and Gan et al. [14], etc. In those methods, the singular perturbation system is decomposed into a slow and a fast subsystem, and this decomposition is possible because feedback control only need slow states. However, in power systems the saturation functions usually contain some fast variables for special goals.

Motivated by this, in this paper, we aim to propose a new method to estimate the stability region for the small-signal dynamical models in power systems based on our recent results [13, 14] and some related singular perturbation theories. We will establish a set of conditions under which the stability region can be decomposed into the Cartesian product of a contractive high-dimensional ellipse and a sufficiently large ball. Furthermore, a method is provided for estimating the stability region with least conservativeness by introducing an LMI-based optimisation model.

This paper is organised as follows. Section 2 provides the model of singular perturbation power systems with saturation non-linearities. Section 3 shows the theories and the algorithm for our method for the stability region estimation. An example is described in Section 4 to show the validity of the suggested method and the conclusion is drawn in Section 5.

Nomenclature: For $x$ and $y \in \mathbb{R}^n$, $x \leq (\geq) y$ means that all elements, say $x_i - y_i$, are (non)negative; symmetric matrix $A \geq 0$ ($A \geq 0$) means that matrix $A$ is (semi)positive definite; Symbol ‘$\prec$‘ means ‘if and only if’; $A$ is a stable matrix if and only if $A$ is a Hurwitz matrix; $\square$ denotes the end of a proof.

2 Singular perturbation model for power systems small-signal stability analysis

In this section, we will formulate the dynamical model for power systems small-signal stability analysis when some saturation non-linearities are considered.

Suppose that the power system under study consists of $n$-buses, $m$-generators and suppose the $m$th generator can be considered to be a infinite generator, take the impedance model for loads, the simple excitation system with additional signal of PSS control for generators. These models, as partly shown in Fig. 1, were also considered in our recent papers [5, 13].

As shown in Fig. 1, we consider that the outputs of excitation and PSS are subject to saturation non-linearities. Owing to the input saturation of the PSS and excitation control systems, the dynamical model of the power systems described above can be expressed as [3, 13]

$$\dot{\delta}_i = \Delta \omega_i$$

$$\omega_0^{-1} \Delta \omega_i = - \sum_{i=1, \ldots, m-1} (K_{i1} \Delta \delta_i - K_{2i} \Delta E_{qi}^i) - D_i \omega_0^{-1} \Delta \omega_i$$

$$T_{d1} \Delta E_{qi}^i = -K_{d1} \Delta \delta_i - K_{d2} \Delta E_{qi}^i + sat(\Delta E_{gd})$$

$$T_{d1} \Delta E_{fd}^i = -K_{d1} \Delta \delta_i - K_{d2} \Delta E_{fd}^i$$

$$T_{w1} \dot{y}_{1i} = -K_{w1} K_i \Delta \delta_i - K_{w1} D_i \omega_0^{-1} \Delta \omega_i$$

$$T_{w1} \dot{y}_{1i} = -K_{w1} K_i \Delta \delta_i - K_{w1} D_i \omega_0^{-1} \Delta \omega_i$$

where expressions (1)–(2) denote the flux decay model of the generators, (3)–(4) is the excitation dynamic model, (5)–(6) is the dynamic model for PSS; in those expressions, subscript ‘$'$’ denotes the variables of the $i$th generator, respectively; $\omega_0$ is the synchronous frequency (50 Hz in our work); $E_{qi}$ is the quadrature axis voltage behind transient reactance; some other variables are partly labelled in Fig. 1, others can be found in [3] or [5, 13]; $sat(\eta) = [sat_1(\eta), sat_2(\eta), \ldots, sat(s)]^T$ and $sat(s) = sign(s) \min\{s_1, |s_2|\}$, $sign(\cdot)$ is the standard sign function and $s$ is the number of all saturation functions. Here we have slightly abused the notation by using $sat(\cdot)$ to denote both the scalar-valued and the vector-valued saturation functions.

As we know, in power systems it is important to analyse the stability of system (1)–(6) corresponding to some given initial states. The fundamental challenge is to analyse the
stability region. In our recent work [5, 13], we gave an approach to estimate the stability region of this kind of systems and used an LMI optimisation method to enlarge the estimated stability region by introducing a fixed shape set made up of initial states. In that approach, we considered all states have the same time scales. However, in a classical power system, the time constant $T_M$ is very small and constant $K_M$ is very large; the time constant $T_M$ is much smaller than $T_H$ in the PSS control system; $H_i$ is much bigger than $T_M$ also. Thus, variables $\Delta E_{i/d}$ and $y_i$ can be considered as the fast variables and the $\Delta E_{q/s}$, $\Delta \delta_i$ should be considered as the slow variables [22]. Hence for a power system with the typical parameters, the time constant $T_M$ is much bigger than $T_H$ in the PSS control system; $H_i$ is much bigger than $T_M$ also. Thus, variables $\Delta E_{i/d}$ and $y_i$ can be considered as the fast variables and the $\Delta E_{q/s}$, $\Delta \delta_i$ should be considered as the slow variables [22].

Therefore some difficulties adhere to singular perturbation dynamical systems exist in the stability region estimation for system (1)–(6). Motivated by this observation, we will use the singular perturbation theories to handle this difficulty resulting from multi-time scales.

For simplicity, in the following analysis we will rewrite system (1)–(6) in a compact form.

Let $K_{TM} = \text{diag}(T_1, K_1/M_1)$, $K_H = \text{diag}(T_w K_0/M_0)$, $M = \text{diag}(2H_I)$, $\Delta \delta = [\Delta \delta_1, \Delta \delta_2, \ldots, \Delta \delta_{m-1}]^T$, $\Delta \omega = [\Delta \omega_1, \Delta \omega_2, \ldots, \Delta \omega_{m-1}]^T$

We can rewrite system (1)–(6) as [13]

$$\dot{\Delta \delta} = \Delta \omega$$ \hspace{1cm} (7)

$$\omega_0^{-1} \dot{M} \Delta \omega = -K_1 \Delta \delta - D \omega_0^{-1} \Delta \omega - K_2 \Delta E_q'$$ \hspace{1cm} (8)

$$T_{d/d} \Delta E_q' = -K_4 \Delta \delta - K_1 \Delta E_q' + \text{sat}(\Delta E_{q/d})$$ \hspace{1cm} (9)

$$T_A \Delta E_{q/d} = -K_A K_5 \Delta \delta - K_A K_6 \Delta E_q' - \Delta E_{q/d} + K_A \text{sat}(y_2)$$ \hspace{1cm} (10)

$$T_w \dot{y}_1 = -K_W K_1 \Delta \delta - K_W D \omega_0^{-1} \Delta \omega - K_W \Delta E_q' - y_1$$ \hspace{1cm} (11)

$$T_2 \dot{y}_2 = -K_{TM} K_1 \Delta \delta - K_{TM} \omega_0^{-1} D \Delta \omega - K_{TM} \Delta E_q' + (I - T_1 T_w^{-1}) y_1 y_2$$ \hspace{1cm} (12)

Further let

$$x = [\Delta \delta^T \Delta \omega^T \Delta E_{q/d}^T \Delta E_q^T]^T, \quad y = \Delta E_{q/d}, \quad z = y_2$$ \hspace{1cm} (13)

Substitute (13) into (7)–(12) and consider the decay speeds of those variables as discussed above, the dynamic model can be rewritten as a popular singular perturbation dynamical system as follows

$$\Sigma: \begin{cases}
    \dot{x} = A_{11} x + B_1 \text{sat}(y) & x(0) = x_0 \\
    \dot{y} = A_{21} x + A_{22} y + B_2 \text{sat}(z) & y(0) = y_0 \\
    \dot{z} = A_{31} x + A_{32} z & z(0) = z_0
\end{cases}$$ \hspace{1cm} (14)

where

$$A_{11} = \begin{bmatrix}
0 & -I_{m-1} & 0 & 0 \\
M^{-1} K_1 & M^{-1} D & M^{-1} K_2 & 0 \\
T_{d/d}^{-1} K_4 & 0 & T_{d/d}^{-1} K_3 & 0 \\
M^{-1} K_1 & M^{-1} K_2 & M^{-1} K_2 & T_w^{-1}
\end{bmatrix}$$

$$B_1 = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}; \quad e = [1, 1, \ldots, 1]^T; \quad A_{22} = -\epsilon T_A^2;$$

$$A_{33} = -\epsilon T_A^{-1} I; \quad B_2 = \epsilon T_A^{-1} K_A; \quad A_{31} = -\epsilon T_A^{-1} [K_{TM} K_1 K_{TM} K_3 (T_1 T_w^{-1} - I)]$$

$$A_{21} = -\epsilon T_A^{-1} K_A [K_5 0 K_6 0];$$

$x(0), y(0)$ and $z(0)$ denote the initial states.

We remark that in the above analysis the load model is considered to be constant impedance model. If we consider the dynamic models such as the motor models for some loads, it is easy to verify that the dynamical model can still be written as the form of expression (14), and the states of motors should be classified into the fast variables since the inertial of a motor is far less than that of a generator. Thus, the structure of $A_{31}$ is still alike that of (14) except that some elements are added in the diagonal position. Moreover, there are many methods for the choice of an appropriate parameter $\epsilon$ [22] in order to formulate the standard singular perturbation model, readers are referred to [22, 23] and the references therein for more information. In this paper, the small parameter $\epsilon$ is chosen to be the minimum among the time scales of fast variables.

Define the stability region $\Omega$ of system (14) as follows

$$\Omega = \{x_0, y_0, z_0 | \varphi(x_0, y_0, z_0) \rightarrow 0\}$$ \hspace{1cm} (15)

where $\varphi(x_0, y_0, z_0)$ denotes the trajectory starting from initial state $(x_0, y_0, z_0)$.

We will try to estimate the stability region of (14) with least conservativeness by considering their special properties, that is, to estimate $\Omega$. To translate this idea into an optimisation problem, in the following analysis, we will introduce the so-called referenced set for initial states similar to those in [13]. Readers are referred to [12] for more information on the notion of referenced set.
3 Optimisation model for stability region estimation

3.1 Problem formulation

Suppose that the referenced set of initial states is a high-dimension ellipse, say \( X_0 = \{ \xi \xi^T P_0 \xi \leq \beta^2 \} \), where \( \xi = [x^T, y^T, z^T]^T \), and \( P_0 > 0 \) is a given positive matrix of appropriate dimensions, \( \beta > 0 \) is a variable to be determined by optimisation. We will enlarge the estimated stability region so that it contains a largest \( X_0 \), that is, obtain the largest \( \beta \) and \( X_0 \subset \omega \) is satisfied meanwhile. Thus, the problem to be solved can be described as:

**Problem 1:** For a given sufficiently small \( \varepsilon > 0 \), how to estimate the stability region of system (14) which can contain a largest set \( X_0 \), and meanwhile solve the difficulties adhere to singular perturbation systems by considering the specialty of this kind of systems, where set \( X_0 \) is a given referenced set defined as

\[
X_0 = \{ \xi \xi^T P_0 \xi \leq \beta^2 \}
\]

Here \( P_0 > 0 \) is a given positive matrix of appropriate dimensions.

Obviously, Problem 1 can be formulated as

\[
\begin{align*}
\beta^* &= \max \beta \\
\text{s.t. } X_0 &= \{ \xi \xi^T P_0 \xi \leq \beta^2 \} \subset \Omega
\end{align*}
\]

In fact, as discussed previously, some traditional methods for estimating the stability region of general dynamical systems with saturation non-linearities can be used to solve this problem, such as the methods in [6–11, 13] and the references therein, but most of them did not consider the multi-time scales. Thus some numerical difficulties may arise. For this reason, we aim to give a new method to solve Problem 1 by using the singular perturbation theories. Firstly, we give two assumptions for system (14):

**Assumption 1:** For system (14), \( A_{11} + B_1 A_{22}^{-1} (A_{21} + B_2 A_{33}) (A_{31}) \), \( A_{22} \) and \( A_{33} \) are Hurwitz (stable) matrices (i.e. their eigenvalues all reside in the left side of the complex plane).

**Assumption 2:** \( \varepsilon \in (0, \varepsilon_0) \) is satisfied where \( \varepsilon_0 > 0 \) is a sufficient small parameter, that is, system (14) can be considered to be a singular perturbation dynamical system.

**Remark 1:** For a real power system, Assumption 1 is trivial. Since feasible excitation and PSS control systems can guarantee the system to have a hyperbolic equilibrium and since \( A_{11} + B_1 A_{22}^{-1} (A_{21} + B_2 A_{33}) (A_{31}) \) is the Jacobian matrix of the degenerate system of (14) [22, 24], \( A_{11} + B_1 A_{22}^{-1} (A_{21} + B_2 A_{33}) (A_{31}) \) is a Hurwitz matrix; matrices \( A_{22} \) and \( A_{33} \) are the diagonal matrices whose elements are negative. Moreover, from the discussion in Section 2, we know that Assumption 2 is usually satisfied for the power systems we described above. In fact, our emphasis is paid to this kind of special dynamical systems and the systems not satisfying Assumption 2 are out of the scope of this paper. Thus, in the following analysis, we will use those assumptions to analyse the stability region for system (14).

3.2 Some lemmas on the time scale properties of singular perturbation system

In this subsection, we will give some definition and properties of singular perturbation system.

**Lemma 1:** Suppose that Assumptions 1–2 are satisfied, and then there exists a unique matrix \( L = [L_1^T, L_2^T]^T \) such that it is a smooth function of \( \varepsilon \) and it satisfies

\[
\begin{align*}
\Gamma_1 &= A_{21} - A_{22} L_1 - B_2 L_2 - \varepsilon L_1 A_\varepsilon = 0 \\
\Gamma_2 &= A_{31} - A_{33} L_2 + \varepsilon L_2 A_\varepsilon = 0
\end{align*}
\]

where \( A_\varepsilon = A_{11} - B_1 L_1 \). Moreover, \( L = [L_1^T, L_2^T]^T \) can be proved by the following iteration

\[
L_{k+1} = \begin{bmatrix} L_1^{k+1} \\ L_2^{k+1} \end{bmatrix} = \begin{bmatrix} A_{22} & B_2 \\ A_{33} \end{bmatrix}^{-1} \begin{bmatrix} A_{21} + \varepsilon L_1^k (A_{11} - B_1 L_1^k) \\ A_{31} + \varepsilon L_2^k (A_{11} - B_1 L_1^k) \end{bmatrix}
\]

and

\[
\begin{align*}
L_2^0 &= \lim_{\varepsilon \to 0} L_2 = A_{33}^{-1} A_{31}, \\
L_1^0 &= \lim_{\varepsilon \to 0} L_1 A_{22}^{-1} (A_{21} + B_2 L_2^0), \\
L^0 &= [L_1^0^T, L_2^0^T]^T
\end{align*}
\]

**Proof:** Since Assumption 1 implies that matrix

\[
\frac{\partial (\Gamma_1, \Gamma_2)}{\partial (L_1, L_2)} = \begin{bmatrix} A_{33} & 0 \\ 0 & A_{33} \end{bmatrix}
\]

is invertible, we conclude matrix solution \( L = [L_1^T, L_2^T]^T \) is a smooth function of \( \varepsilon \) from the implicit function theorem [24]. Iteration (18) can be concluded by Lemma 2.2 (in [22], p. 53).

Define a new coordinate as

\[
\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ L_1 & I & 0 \\ L_2 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]
And rewrite system (14) in the new coordinate, we have
\[ \begin{align*}
\dot{x} &= A_1 \dot{x} + B_1 q_2(x, \tilde{y}) \\
\dot{\tilde{y}} &= A_{22} \tilde{y} + B_2 q_2(x, \tilde{y}) - e L_1 B_1 q_2(x, \tilde{y}) \\
\dot{\tilde{z}} &= A_{33} \tilde{z} - e L_2 B_1 q_2(\tilde{x}) \tilde{y}
\end{align*} \tag{20} \]

where \( q_2(x, \tilde{z}) = sat(\tilde{z} + L_2 \tilde{x}) - L_2 \tilde{x}, \quad q_3(\tilde{x}, \tilde{y}) = sat(\tilde{y} + L_1 \tilde{x}) - L_1 \tilde{x} \).

Define the following sets
\[ F_1 = \{ x | -e \leq Lx \leq e \} \tag{22} \]
\[ F_2 = \{ x | -e \leq Lx \leq e \} \tag{23} \]
\[ E(P_a, 1) = \{ x | x^T P_a x \leq 1 \} \tag{24} \]
\[ \Omega_0(r) = \{ \| y \| \leq r \} \tag{25} \]
\[ \Omega_r(r) = \{ \| z \| \leq r \} \tag{26} \]

where \( e = [1, 1, \ldots, 1]^T \) is a vector whose elements are all 1, the inequalities is on the basis of element by element, \( \sigma \in (0, 1) \) is a given constant which will be further discussed later; \( r > 0 \) is a given constant.

Define an adjoint system as
\[ \Sigma_2 \dot{x}_2 = A_2 x_2 \tag{27} \]

Let the trajectories of system \( \Sigma_2 \) and system (20) are denoted by \( x_2(t, e) \) and \( \tilde{x}(t, e) \), respectively. Next we give some lemmas on the relationship between them.

**Lemma 2:** Suppose Assumptions 1–2 are satisfied, then \( A_s = A_{11} + B_1 L_1 \) is a stable matrix for all \( e \in [0, e_0] \), that is, there exist \( \alpha_0 > 0 \) and \( \lambda_0 > 0 \) independent of \( e \) such that
\[ \| e^{A_s t} \| \leq e^{-\lambda_0 t}, \quad \forall t \in [0, +\infty) \times (0, e_0) \tag{28} \]

*Proof:* Lemma 1 and Assumption 1 implies that \( A_{11} = A_{11} + B_1 L_1 \) is a stable matrix. Therefore for sufficiently small \( e_0 \) (Assumption 2), there exist \( \alpha_0 > 0 \) and \( \lambda_0 > 0 \) such that [24]
\[ \| e^{A_s t} \| \leq e^{-\lambda_0 t}, \quad \forall t \in [0, +\infty) \times (0, e_0) \tag{29} \]

**Lemma 3:** Consider system (20)–(21). Suppose Assumptions 1–2 are satisfied, then we have if \( \tilde{x}(t, e) \in F_1 \) satisfies for \( \forall t \in [0, T] \) where \( T > 0 \), then there exist \( \gamma_1 > 0 \) and \( \alpha_1 > 0 \) independent of \( e \) such that
\[ \left\| \begin{bmatrix} \tilde{x}(t, e) \\ \tilde{z}(t, e) \end{bmatrix} \right\| \leq \alpha_1 \left\| \begin{bmatrix} \tilde{x}(0) \\ \tilde{z}(0) \end{bmatrix} \right\| e^{-\lambda_1 t/e}, \quad \forall t \in [0, T] \tag{30} \]

where \( \tilde{x}(t, e) \) and \( \tilde{z}(t, e) \) denote the trajectories of system (21).

**Proof:** For \( \forall a, b \in R, \) since \( |a| \geq 1 \) can lead to \( |a t + b - a| \leq |b| \) [25], there exist \( \alpha_2 > 0 \) and \( \alpha_3 > 0 \) such that if \( (\tilde{x}(t, e) \in F_1 \) is satisfied for \( \forall t \in [0, T] \), the inequalities
\[ \| q_1(\tilde{x}, \tilde{z}) \| = \| sat(\tilde{z} + L_2 \tilde{x}) - L_2 \tilde{x} \| = \alpha_2 \| \tilde{z} \| \tag{31} \]
\[ \| q_2(\tilde{y}, \tilde{y}) \| = \| sat(\tilde{y} + L_1 \tilde{x}) - L_1 \tilde{x} \| = \alpha_3 \| \tilde{y} \| \tag{32} \]

are satisfied uniformly for all sufficiently small \( e \in (0, e_0) \).

We consider the system as
\[ \begin{align*}
\dot{y} &= A_{22} y + B_2 q_1(x, \tilde{z}) \\
\dot{\tilde{z}} &= A_{33} \tilde{z} \tag{33} \]

Obviously, system (33) is a globally exponentially stable system since \( A_{22} \) and \( A_{33} \) are stable matrices and since \( q_1(x, \tilde{z}) \) does not contain \( \tilde{y} \) explicitly when \( \tilde{x}(t, e) \in F_1 \) is satisfied. We further view system (21) as a perturbation system of (33) (i.e. the perturbation terms are \( e L_2 B_1 q_2(x, \tilde{y}) \) and \( e L_2 B_1 q_2(\tilde{x}, \tilde{y}) \), so from [24] (Lemma 5.1, p. 205) it can be easily verified that system (21) is also an exponentially stable system by considering (31)–(32), thus there exist \( \gamma_1 > 0 \) and \( \alpha_1 > 0 \) such that the trajectories of system satisfy (30) uniformly for all \( e \in (0, e_0) \).

**Lemma 4:** Consider systems (20)–(21) and (27). Suppose Assumptions 1–2 are satisfied, then we have the following conclusions:

1. If \( ((\tilde{x}(0), \tilde{z}(0)) \in \Omega_0(r) \times \Omega_r(r) \) and \( \tilde{x}(t, e) \in F_1 \) is satisfied for \( \forall t \in [0, T] \) where \( T > 0 \). Then there is
\[ \tilde{x}(t, e) = x_2(t, e) + O(e), \quad \forall t \in [0, T] \times (0, e_0) \tag{34} \]

2. If \( ((\tilde{x}(0), \tilde{z}(0)) \in \Omega_0(r) \times \Omega_r(r) \) and \( x_2(t, e) \in F_1 \) is satisfied for \( \forall t \in [0, +\infty) \), then there are \( \tilde{x}(t, e) \in F_1 \) and \( \forall t \in [0, +\infty) \times (0, e_0) \)
\[ \tilde{x}(t, e) = x_2(t, e) + O(e), \quad \forall t \in [0, +\infty) \times (0, e_0) \tag{35} \]

Moreover,
\[ \lim_{t \to \infty} \| \tilde{x}(t, e) - x_2(t, e) \| = 0 \tag{36} \]

**Proof:** We prove those conclusions one by one.

**Proof of conclusion 1:** From expression (20), we have
\[ \tilde{x}(t, e) = e^{A_{22} t} \tilde{x}(0) + \int_0^t e^{A_{22} (t-s)} B_2 q_1(\tilde{x}(s, e), \tilde{z}(s, e)) \, ds \]
\[ = \tilde{x}_2(t, e) + \int_0^t e^{A_{22} (t-s)} B_2 q_1(\tilde{x}(s, e), \tilde{z}(s, e)) \, ds \tag{37} \]
And from (32), we know that if $\bar{x}(t, \varepsilon) \in F_1$ is satisfied there is

$$
\|\bar{x}(t, \varepsilon) - \bar{x}_j(t, \varepsilon)\| = \left\| \int_0^t e^{A(t\sigma - \sigma)} B_1 q_2(\bar{x}(t, \varepsilon), \bar{y}(t, \varepsilon)) \, d\sigma \right\|
$$

From (29) in Lemma 2 and from (30) in Lemma 3, for sufficiently small $\varepsilon_0 > 0$ we have

$$
\|\bar{x}(t, \varepsilon) - \bar{x}_j(t, \varepsilon)\| \leq \sqrt{2} \alpha_3 \|B_1\| \int_0^t e^{-\lambda(t - \sigma)} e^{-\lambda_1/\varepsilon} \, ds
$$

From (29) in Lemma 2 and from (30) in Lemma 3, for sufficiently small $\varepsilon_0 > 0$ we have

$$
\|\bar{x}(t, \varepsilon) - \bar{x}_j(t, \varepsilon)\| \leq \sqrt{2} \alpha_3 \|B_1\| \int_0^t e^{-\lambda(t - \sigma)} e^{-\lambda_1/\varepsilon} \, ds
$$

Thus if $\bar{x}(t, \varepsilon) \in F_1$ is satisfied, we can choose an appropriate $M_0$ independent of $\varepsilon$ and $t$ such that

$$
\|\bar{x}(t, \varepsilon) - \bar{x}_j(t, \varepsilon)\| \leq M_0 \varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0), \quad \forall t \in [0, T]
$$

is satisfied. \hfill $\square$

**Proof of conclusion 2:** We use a contradiction argument. Since the trajectories $x_i(t, \varepsilon)$ and $\bar{x}(t, \varepsilon)$ have the same initial states $x_0$, and since $F_0 \subset F_1$ from the definitions of $F_0$ and $F_1$ (in (22) and (23), respectively), if we suppose that the result is not correct, then there exists a $T_1 \in (0, +\infty)$ such that $x_i(T_1, \varepsilon) \in F_0$ and $\bar{x}(T_1, \varepsilon) \in \partial F_1$ are satisfied, where $\partial F_1$ denotes the boundary of set $F_1$. From the definitions of $F_0$ and $F_1$, there exists a subspace $i$ such that $|L_i \bar{x}(T_1, \varepsilon) = 1$ and $|L_i x_i(T_1, \varepsilon)| \leq \sigma$ are satisfied, where $L_i$ denotes the $i$th line of $L$. Therefore

$$
|L_i(\bar{x}(T_1, \varepsilon) - x_i(T_1, \varepsilon))| \geq |L_i \bar{x}(T_1, \varepsilon)| - |L_i x_i(T_1, \varepsilon)| \geq 1 - \sigma
$$

that is, for sufficiently small $\varepsilon_0 > 0$ there exists a constant $\beta > 0$ which is dependent of the given constant $\sigma$ but not independent of $\varepsilon$ and $T_1$ such that we have

$$
\|\bar{x}(T_1, \varepsilon) - x_i(T_1, \varepsilon)\| > \beta(1 - \sigma), \quad \forall \varepsilon \in (0, \varepsilon_0)
$$

However, $\bar{x}(t, \varepsilon) \in F_1$ is satisfied for $\forall t \in [0, T_1]$, so from (38) in conclusion 1 we have

$$
\|\bar{x}(t, \varepsilon) - \bar{x}_j(t, \varepsilon)\| \leq M_0 \varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0)
$$

Then, since $M_0$ is a constant independent of $\varepsilon$ and $t$, there exists a $\varepsilon_0^\sigma = \min\{\varepsilon_0^\sigma, \varepsilon_0^\sigma, \beta(1 - \sigma)/M_0\}$ such that expressions (39) and (40) are contradictory.

Therefore if Assumptions 1–2 are satisfied (there exists a sufficiently small $\varepsilon_0$), there is

$$
\bar{x}(t, \varepsilon) = x_i(t, \varepsilon) + O(\varepsilon), \quad \forall \varepsilon \in [0, +\infty) \times (0, \varepsilon_0)
$$

Moreover, from expression (37) we know that $M$ will converge to zero as $t \to +\infty$, so expression (36) is satisfied. Therefore we can finish the proof.

**Remark 2:**

1. In the Tikhonov theorem [22, 24], the manifold defined as $[y^T, z^T]^T = L^T x$ is called a ‘quasi-steady-state’ or ‘slow manifold’ ($\varepsilon = 0$), and system (21) is called the boundary-layer system when the slow variables and $t$ are fixed. Moreover, the Tikhonov theorem addressed some results similar to Lemma 3 and Lemma 4, but their results usually are given for a finite time interval and the proof is also slightly different from our results (in fact, we use the skill in [22], so the adjoint system and the quasi-steady-state are slightly different also). It is the special structure of origin system (14) and the special property of saturation (i.e. (31) and (32)) that lead to the satisfaction of the important results in an infinite time interval. These results can be considered to be the extension of Tikhonov theorem in the singular perturbation systems with saturation nonlinearities by considering some special properties in power systems.

2. It is worth noting that the special property of saturation (i.e. (31) and (32)) contribute to reduce the impact of fast variables on slow variables. In other words, though a trajectory make the saturation functions reach their bounds, the trajectory may still converge to the stability region. Therefore the estimated region may exist in the nonlinear region of saturation and this result is clearly better than the method provided in [13].

3. In Lemma 4, for a fixed $r$ we can always find an appropriate $\varepsilon_0 > 0$ such that for $\forall \varepsilon \in (0, \varepsilon_0)$ the discrepancies between the trajectories of adjoint system (27) and origin system (20) will converge to zero. Hence in power systems, if the given parameter $\varepsilon$ is small enough (Assumption 2 is satisfied), we can choose a large $r$ in order to simplify the stability region estimation as shown in next subsection.

**3.3 LMI-based optimisation method:** Based on those lemmas provided in Section 3.2, we can obtain the following theorem.

**Theorem 1:** For systems (20)–(21) and (27), suppose that the following conditions are satisfied:

1. Assumptions 1–2 are satisfied.
2. There exists a positive definite matrix $P_a$ of appropriate dimension such that it satisfies

$$A_d^TP_a + P_aA_d < 0$$

3. $\Omega_a = E(P_a, 1)$ is a subset of set $F_a$, where set $E(P_a, 1)$ and $F_a$ is defined in section 3.2.

4. $r > 0$ is sufficiently large constant and $\sigma \in (0, 1)$ is given constant (in the definition of $F_a$).

Then Cartesian product set $\Omega_x \times \Omega_y \times \Omega_z$ is a subset of the stability region of system (20)–(21), that is, set $\Omega_x \times \Omega_y \times \Omega_z$ is one of the results of the stability region estimation.

Proof: Arbitrarily choose an initial state in a $\Omega_x \times \Omega_y \times \Omega_z$, say $(x_0, y_0, z_0)$. From the given condition 2 and condition 3, we know that $\Omega_y = E(P_a,1) \subset F_a$ is an invariant set for the adjoint system (27) [12], so the trajectory of adjoint system (27) satisfies $x(t,e) \in \Omega_y \subset F_a$ for $\forall (t,e) \in [0, +\infty) \times (0, e_0)$. From Lemma 1 the adjoint system is a globally exponentially stable system, so

$$\|x(t,e)\| \to 0 \text{ as } t \to \infty$$

(41)

By Lemma 4, we obtain

$$\tilde{x}(t,e) \in F_1, \quad \forall (t,e) \in [0, +\infty) \times (0, e_0)$$

(42)

and

$$\lim_{t \to \infty} \|\tilde{x}(t,e) - x(t,e)\| = 0$$

(43)

Thus, by Lemma 3 and expression (42), for $\forall e \in (0, e_0)$ we have

$$\|\tilde{x}(t,e)\| \to 0 \text{ and } \|\tilde{z}(t,e)\| \to 0 \text{ as } t \to \infty$$

and by expressions (41) and (43), for $\forall e \in (0, e_0)$ we have

$$\|\tilde{x}(t,e)\| \to 0 \text{ as } t \to \infty$$

Therefore the trajectory of system (20)–(21) starting from a $\Omega_x \times \Omega_y \times \Omega_z$, will converge to the origin point as $t$ approaches infinite, that is, set $\Omega_x \times \Omega_y \times \Omega_z$ is a subset of the stability region.

Obviously, Theorem 1 shows that we can estimate the stability region of system (20)–(21) by its degenerate system (27). Therefore in order to estimate the stability region of system (20)–(21), we only need to find set $\Omega_x = E(P_a,1) \subset F_a$ since $\Omega_y$ and $\Omega_z$ can choose an appropriate constant $r$. (The maximum of $r$ is relative to $e$, but if $e$ is sufficiently small we always can find the $r$ as we want.) Namely, we only need find a positive definite matrix $P_a$ satisfying conditions 2 and 3. Moreover, from [12, 13] we know that by the Schur complements of matrices, condition 3 can be easily changed to an LMI as follows

$$\Omega_a = E(P_a, 1) \subset F_a[x - \sigma e \leq Lx \leq \sigma e]$$

$$\iff \sigma^2 - I - P_a^{-1}I^T \geq 0$$

$$\iff \left[ \begin{array}{cc} \sigma^2 & I \\ I^T & P_a \end{array} \right] \geq 0, \quad i = 1, 2, \ldots, s$$

(44)

where $I$ is the $i$th line of matrix $L$.

Combining (16) and (44), we can obtain the following optimisation model with the variable of $P_a$ to solve Problem 1

$$\beta^* = \max_{P_a>0} \beta$$

s.t.

$$X_0 = \{\xi|\xi^TP_a \xi \leq \beta^2\} \subset \Omega_x \times \Omega_y(r) \times \Omega_z(r)$$

Moreover, from [12, 13] we know that by the Schur complements of matrices, condition 3 can be easily changed to an LMI as follows

$$\Omega_a = E(P_a, 1) \subset F_a[x - \sigma e \leq Lx \leq \sigma e]$$

$$\iff \sigma^2 - I - P_a^{-1}I^T \geq 0$$

$$\iff \left[ \begin{array}{cc} \sigma^2 & I \\ I^T & P_a \end{array} \right] \geq 0, \quad i = 1, 2, \ldots, s$$

(44)

Remark 3: In this optimisation, referenced set $X_0 = \{\xi|\xi^TP_a \xi \leq \beta^2\}$ is expressed in the new coordinate $(\tilde{x}, \tilde{y}, \tilde{z})$ defined as (19), so the expression of $X_0$ needs to be changed and so does the $P_a$. But for brevity, we still use $P_a$ to denote this referenced set.

Optimisation problem (45) is alike an LMI optimisation except for the first constraint. So next we change the first constraint to an LMI also. Since $r$ is a sufficiently large constant, $\Omega_x \times \Omega_y(r) \times \Omega_z(r)$ along the direction of axis $y$ and $z$ are sufficiently large. So $X_0 \subset \Omega_x \times \Omega_y(r) \times \Omega_z(r)$ is satisfied if and only if the projection of $X_0$ on coordinate $x$ is a subset of set $\Omega_x$. To obtain the projection of $X_0$ on axis $x$, we give a lemma:

Lemma 5: Suppose a high-dimensional ellipse is denoted by

$$E(P, \rho) = \left\{ \begin{array}{c} \xi_1 \\ \xi_2 \\ \xi_3 \\ P_{11} \xi_1 + P_{12} \xi_2 + P_{13} \xi_3 \leq \rho \end{array} \right\}$$

whose projections on axis $\xi_1$ and $\xi_2$ are denoted by $E_{\xi_1}(P, \rho)$ and $E_{\xi_2}(P, \rho)$, respectively. Then both $E_{\xi_1}(P, \rho)$ and $E_{\xi_2}(P, \rho)$ are ellipse. Moreover, they can be expressed as $E_{\xi_1} = \{x|\xi_1^TP_{11}x + \xi_1^TP_{12}x_2 + \xi_1^TP_{13}x_3 \leq \rho\}$ and $E_{\xi_2} = \{x|\xi_2^TP_{22}x_2 + \xi_2^TP_{21}x_1 + \xi_2^TP_{23}x_3 \leq \rho\}$, respectively.

Proof: Since the expression of $E(P, \rho)$ can be expressed as

$$V(\xi_1, \xi_2) = \xi_1^TP_{11}\xi_1 + 2\xi_1^TP_{12}\xi_2 + \xi_2^TP_{22}\xi_2 \leq \rho$$

(46)

The projection of $E(P, \rho)$ on axis $\xi_1$ is a set satisfying: for $\forall \xi_1 \in E_{\xi_1}(P, \rho)$, in the Euclidean spaces there exists a
solution \( \xi_2 \) satisfying (46). Moreover, for \( \forall \xi_1 \in E_\xi(P, \rho) \), there exists a \( \xi_2 \) satisfying (46) if and only if the minimum value of \( V(\xi_1, \xi_2) \) is less than \( \rho \), that is

\[
\min_{\xi_2} V(\xi_1, \xi_2) = V(\xi_1, \xi_2)|_{\xi_2 = -P^{-1}_{12}P^{12}_{12}\xi_1} = \xi_1^T(P_{11} - P_{12}P^{12}_{22}P^{12}_{12})\xi_1 \leq \rho
\]

Therefore, the projection of \( E(P, \rho) \) on axis \( \xi_1 \) satisfies \( E_\xi(P, \rho) = \{ \xi_1 : \xi_1^T(P_{11} - P_{12}P^{12}_{22}P^{12}_{12})\xi_1 \leq \rho \} \). By the Schur complements of matrices expression (47), we can prove that \( E_\xi(P, 1) = \{ \xi_1 : \xi_1^T(P_{22} - P_{12}P^{12}_{12})\xi_1 \leq \rho \} \) is an ellipse. Similarly, by Combing (45) and (48), Problem 1 can be solved by an LMI optimisation problem as shown in (50), and thus estimating the stability region of system (14) is simplified comparing to some traditional methods as shown in [12, 13], and so on.

**Remarks 4:**

1. Since optimisation problem (50) is of order \( n_x \)-dimension (dimension of \( x \)) and matrix \( A_x \) is not a singular matrix (in fact, matrix \( A_x \) is the Jacobian matrix of the degenerate adjoint system), our method can avoid the ill-condition adherence to singular perturbation systems and can reduce calculation burden.

2. As proved in Theorem 1, the maximum of feasible \( \sigma \) (in the definition of \( F_{\sigma} \)) is relative to \( \varepsilon \), so sometimes we need to change \( \sigma \). However, the change of \( \sigma \) lead to recalculating the optimal solution of (50). Fortunately, when \( \sigma > 0 \), the following conclusion about (50) can help avoid extra calculation.

**Lemma 6:** For the optimisation problem (50), the optimal solution, say, \( \beta^* \) (or \( \gamma^* \)), is in proportion to parameter \( \sigma \). Namely, if \( \sigma \) is changed from \( \sigma_1 \) to \( \sigma_2 = \rho \sigma_1 \), then the optimal solution \( \beta^* \) is also changed from \( \beta_1^* \) to \( \beta_2^* = \rho \beta_1^* \), where \( \rho > 0 \) is a constant and the \( \beta_1^* \) and \( \beta_2^* \) are the optimal solutions when \( \sigma = \sigma_1 \) and \( \sigma = \sigma_2 \), respectively.

**Proof:** Let \( \gamma_1^* \) and \( P^{a1}_{a1} \) denote the optimal decision variables of optimisation problem (50) with \( \sigma = \sigma_1 \), and let \( \gamma_2^* \) and \( P^{a2}_{a2} \) denote the optimal decision variables with \( \sigma = \sigma_2 \).

Since \( \gamma_1^* \) is an optimal solution of optimisation problem (50) with \( \sigma = \sigma_1 \), we have

\[
\begin{align*}
\gamma_1^*P^{a0}_{a0} - P^{a1}_{a1} > & 0 \\
\sigma_1^2 - I^TP^{a1}_{a1}I^T > & 0, \quad i = 1, 2, \ldots, s \\
A^T_{a1}P^{a1}_{a1} + P^{a1}_{a1}A_1 < & 0
\end{align*}
\]

Multiply both sides of the first and the third expressions with \( \rho^{-2} \), and the second expression with \( \rho^2 \), then there is

\[
\begin{align*}
\left( \rho^{-2}\gamma_1^* \right)P^{a0}_{a0} - \left( \rho^{-2}P^{a1}_{a1} \right) > & 0 \\
\left( \rho\sigma_1 \right)^2 - I^T\left( \rho^{-2}P^{a1}_{a1} \right)^{-1}I^T > & 0, \quad i = 1, 2, \ldots, s \\
A^T_{a1}\left( \rho^{-2}P^{a1}_{a1} \right) + \left( \rho^{-2}P^{a1}_{a1} \right)A_1 < & 0
\end{align*}
\]

From (51), it is evident that \( \rho^{-2}\gamma_1^* \) and \( \rho^{-2}P^{a1}_{a1} \) is a group of feasible solution of optimisation (50) when \( \sigma = \rho \sigma_1 \). However, \( \gamma_2^* \) is the optimal value of (50) when \( \sigma = \sigma_2 = \rho \sigma_1 \), so \( \rho^{-2}\gamma_1^* \) and \( \rho^{-2}P^{a1}_{a1} \) satisfy

\[
\rho^{-2}\gamma_1^* \geq \gamma_2^*
\]
Similarly, multiply both sides of the first and the third expressions (52) and (55) imply that

$$\begin{align*}
\gamma_2^2 P_2^0 - P_2^a &> 0 \\
\sigma_2^2 - I(P_2^a)^{-2} P_2^a &\geq 0, \quad \text{i} = 1, 2, \ldots, s \\
A_2^s (p^2 P_2^a) + P_2^a A_a &< 0 
\end{align*}$$

(53)

From (54), it is also evident that \( \rho^2 \gamma_2^* \) and \( \rho^2 P_2^a \) are a group of feasible variables of (50) when \( \sigma = \rho^{-1} \sigma_2 = \sigma_1 \). However, \( \gamma_1^* \) is the optimal value of (50) with \( \sigma = \sigma_1 \), so \( \rho^2 \gamma_2^* \) and \( \rho^2 P_2^a \) satisfy

$$\rho^2 \gamma_2^* \geq \gamma_1^*$$

(55)

Expressions (52) and (55) imply that \( \gamma_1^* = \rho^2 \gamma_2^* \). By (49), we can obtain \( \beta_2^* = \rho \beta_1^* \).

Lemma 6 shows that the optimal solution \( \beta^* \) is in proportion to \( \sigma \). Therefore we can choose an arbitrary \( \sigma > 0 \) when handling optimisation problem (50), and use the linear relationship to obtain the different \( \beta^* \) corresponding to different \( \sigma \), avoiding the redundant calculations.

Summarising previous analysis, Problem 1 can be handled via solving optimisation problem (50). The algorithm is as follows (Algorithm 1):

**Step 1**: Set up the model as (1)–(6) for power systems.

**Step 2**: Choose the appropriate parameter \( \varepsilon \), the fast and the slow variables in power systems, and then set up the singular perturbation dynamical system as (14).

**Step 3**: Calculate variables such as \( A_a \) and \( L \) via iteration (18), choose an appropriate experienced constant \( \sigma \) and introduce the referenced set of initial states \( X_0 \), then formulate LMI optimisation model (45).

**Step 4**: Solve LMI optimisation problem (45) to obtain the best estimation according to the introduced referenced set \( X_0^* \).

### 4 Simulation results in power systems

In this section, we will show that the optimisation results based on the degenerate system are valid by using a test power system of single machine against infinite bus (SMIB). The parameters of this power system are partly shown in Fig. 2, the controller parameters are

\[
H = 3.5, \quad U_{pss} = -U_{pss} = 2, \quad E_{f_{\text{max}}} = -E_{f_{\text{min}}} = 6.2, \\
T_{15} = 1.4, \quad T_{10} = 0.02, \quad K_c = 9.0, \quad T_{1} = 0.154, \\
T_2 = 0.033, \quad K_d = 10, \quad X_d = 1.81, \quad X_q = 1.76, \\
X_p = 0.3, \quad T_{dss} = 8.0, \quad D = 0
\]

Choose \( \varepsilon = T_{15} = 0.02, \ \sigma = 0.95 \) and the referenced set of initial states is a unit-ball, that is, \( P_0 \) is an identity matrix. By time domain simulation, we observe that the origin dynamical system is a singular perturbation systems, as shown in Fig. 3 and Table 1. In this figure, we observe that the decay speeds of \( E_{f_{\text{ss}}} \) and \( Y_{2(2)} \) are far more larger than that of the \( \Delta \delta \), those fast variables converge to the corresponding slow manifolds which are determined by slow variables as shown in Remark 2), \( E_{f_{\text{ss}}} \) and \( Y_{2(2)} \) can considered to be the fast variables, of the origin system. Eigenvalues in Table 1 also illustrate this result since the condition-number is very large.

By Algorithm 1 we obtain the optimal solution \( \beta^* = 0.182 \). The corresponding optimal estimation of the stability region and the maximum referenced set \( X_0 \) are shown in Fig. 4. Fig. 4 also plots the time history of a trajectory (denoted by the dot curve) starting from the estimated stability region. Corresponding to this trajectory, the values of saturation function are shown in Fig. 5. From
Figs. 4 and 5, we note that the state trajectory is not smooth; this is because the trajectory reaches the bound of saturation functions, as shown in Fig. 5. Therefore the estimated stability region does not completely reside in the linear region of the saturation functions. This result is in accordance with what we discussed in Section 3.

To show that our suggested method can alleviate the singularity resulting from the small parameter $\varepsilon$, Table 1 shows the eigenvalues of the Jacobian matrices of the adjoint system and of the origin system, respectively. Fig. 4 shows the fast variables of the origin system in time domain. From Table 1, we observe that the condition number corresponding to the adjoint system is much lower than that of the primary system. Since the suggested method is based on the adjoint system, it can alleviate the singularity to some extent.

### 5 Conclusion

This paper provided a singular perturbation model for small-signal stability analysis for power systems with detailed excitation and PSS controllers. The saturation non-linearities existing in the controllers were considered in the singular perturbation model. Furthermore, an LMI-based method was provided for estimating the stability region of this kind of systems with least conservativeness by introducing an LMI optimisation model. Compared with some traditional methods, the suggested method can alleviate some numerical problems adhere to singular perturbation systems, such as the ill-condition, which are usually encountered in power systems stability analysis. Simulation results indicated the effectiveness of our method by using a test power system with saturated PSS controllers.

### 6 Acknowledgments

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### 7 References


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**Table 1** Eigenvalues of the origin system and the adjoint system

<table>
<thead>
<tr>
<th>Origin system</th>
<th>Adjoint system</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-49.2893, -30.0506$</td>
<td>$-$</td>
</tr>
<tr>
<td>$-0.1537 \pm 6.5329i$</td>
<td>$-0.1535 \pm 6.5335i$</td>
</tr>
<tr>
<td>$-0.7680, -1.1177$</td>
<td>$-0.7680, -1.1175$</td>
</tr>
</tbody>
</table>

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**Figure 4** Results of stability region estimation

**Figure 5** Values of saturation function along the state trajectories

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